

Last time: Basic notations + terminology.

## Linear Systems of Equations

Defn: Let  $x_1, x_2, \dots, x_n$  be variable symbols (or variables).

A linear combination of these variables is any sum of form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

where  $a_1, a_2, \dots, a_n$  are constants (i.e. coefficients).

NB: Constants are real numbers.

A linear equation is an equation  $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$

where  $a_i$ 's and  $b$  are all constants. A linear system of equations (or linear system) is a collection of linear equations.

$$\begin{cases} a_{1,1} x_1 + a_{1,2} x_2 + \dots + a_{1,n} x_n = b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 + \dots + a_{2,n} x_n = b_2 \\ \vdots \\ a_{m,1} x_1 + a_{m,2} x_2 + \dots + a_{m,n} x_n = b_m \end{cases}$$

NB: This is an  $m \times n$  system, or a system with  $m$  equations in  $n$  unknowns.

Ex: The system  $\begin{cases} x - y + 2z = 0 \\ 3x + 0y + 4z = 4 \\ y + 0x - 2z = 2 \end{cases}$  is linear

$$x \leftrightarrow x_1, \quad y \leftrightarrow x_2, \quad z \leftrightarrow x_3$$

Non Ex: The system  $\begin{cases} x^2 + y^2 = 4 \\ -y + x = 3 \end{cases}$  is not linear

Defn: A solution to an  $m \times n$  linear system is an  $n$ -tuple (or vector) of constants satisfying all equations simultaneously.

Goal: Given a linear system, compute all solutions

Ex: Solve 
$$\begin{cases} x - y + 2z = 0 \\ 3x + 4z = 4 \\ y - 2z = 2 \end{cases}$$

Sol: If  $(x, y, z)$  is a solution to this system:

$$\begin{cases} x - y + 2z = 0 \\ 3x + 4z = 4 \\ y - 2z = 2 \end{cases} \xrightarrow{E_1 + E_3 \rightarrow E_1} \begin{cases} x = 2 \\ 3x + 4z = 4 \\ y - 2z = 2 \end{cases}$$

$$\xrightarrow{E_2 - 3E_1 \rightarrow E_2} \begin{cases} x = 2 \\ 4z = -2 \\ y - 2z = 2 \end{cases} \xrightarrow{\frac{1}{4}E_2 \rightarrow E_2} \begin{cases} x = 2 \\ z = -\frac{1}{2} \\ y - 2z = 2 \end{cases}$$

$$\xrightarrow{E_3 + 2E_2 \rightarrow E_3} \begin{cases} x = 2 \\ z = -\frac{1}{2} \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1 \\ z = -\frac{1}{2} \end{cases}$$

$\therefore$  The system has solution  $\begin{bmatrix} 2 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

NB: The above method is Gaussian elimination.

This method always solves a linear system.

IDEA: Systematic elimination of variables...

NB: Every linear system can be solved using the

\* following three operations:

① Swap two rows

② Multiply a row by a nonzero constant.

③ Add two rows and replace one of them with the result.

These are the elementary (row) operations.

Ex: Solve the system

$$\begin{cases} 2x - 2y + z = 0 \\ 4y + z = 20 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$$

NB: this  $4 \times 3$  system is "overdetermined" because it has more equations than variables.

$$\begin{cases} 2x - 2y + z = 0 \\ 4y + z = 20 \\ x + z = 5 \\ x + y - z = 10 \end{cases} \xrightarrow{E1 \leftrightarrow E3} \begin{cases} x + z = 5 \\ 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + y - z = 10 \end{cases}$$

$$\begin{cases} E3 - 2E1 \rightarrow E3 \\ E4 - E1 \rightarrow E4 \end{cases} \begin{cases} x + z = 5 \\ 4y + z = 20 \\ -2y - z = -10 \\ y - 2z = 5 \end{cases} \xrightarrow{\begin{matrix} E4 \leftrightarrow E2 \\ -E3 \rightarrow E3 \end{matrix}} \begin{cases} x + z = 5 \\ y - 2z = 5 \\ +2y + z = +10 \\ 4y + z = 20 \end{cases}$$

$$\begin{cases} E3 - 2E2 \rightarrow E3 \\ E4 - 4E2 \rightarrow E4 \end{cases} \begin{cases} x + z = 5 \\ y - 2z = 5 \\ 5z = 0 \\ 9z = 0 \end{cases} \xrightarrow{\begin{matrix} \frac{1}{5}E3 \rightarrow E3 \\ \frac{1}{9}E4 \rightarrow E4 \end{matrix}} \begin{cases} x + z = 5 \\ y - 2z = 5 \\ z = 0 \\ z = 0 \end{cases}$$

$$\begin{cases} E4 - E3 \rightarrow E4 \\ E2 + 2E3 \rightarrow E2 \\ E1 - E3 \rightarrow E1 \end{cases} \begin{cases} x = 5 \\ y = 5 \\ z = 0 \\ 0 = 0 \end{cases} \longrightarrow \begin{cases} x = 5 \\ y = 5 \\ z = 0 \end{cases} \quad \leftarrow$$

Hence  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$  is the unique solution to this system  $\square$

Ex: Solve  $\begin{cases} x - y + z = 2 \\ x + y - z = -1 \\ 3x + y - z = 1 \end{cases}$

Sol: Applying Gaussian elimination:

$$\begin{cases} x - y + z = 2 \\ x + y - z = -1 \\ 3x + y - z = 1 \end{cases} \xrightarrow{\begin{matrix} E2 - E1 \rightarrow E2 \\ E3 - 3E1 \rightarrow E3 \end{matrix}} \begin{cases} x - y + z = 2 \\ 2y - 2z = -3 \\ 4y - 4z = -5 \end{cases}$$

$$\xrightarrow{E3-2E2 \rightarrow E3} \begin{cases} x - y + z = 2 \\ 2y - 2z = -3 \\ 0 = 1 \end{cases} \leftarrow \text{Contradiction.}$$

↑ Never true!

$$\begin{cases} x - y + z = 2 \\ 2y - 2z = -3 \\ 4y - 4z = -5 \end{cases}$$

Hence this system has no solutions!

I.e. its solution set is empty.

□

Ex: Solve the system

$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

Sol:

$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases} \xrightarrow{\substack{E4-2E1 \rightarrow E4 \\ E3-E1 \rightarrow E3}} \begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ x - 2z - 2w = -5 \\ y - w = -1 \end{cases}$$

$$\xrightarrow{\substack{E1-2E3 \rightarrow E1 \\ E4-E2 \rightarrow E4}} \begin{cases} 5z + 5w = 15 \\ y - w = -1 \\ x - 2z - 2w = -5 \\ 0 = 0 \end{cases}$$

$$\xrightarrow{\substack{\frac{1}{5}E1 \rightarrow E1 \\ 2E1 + E3 \rightarrow E3}} \begin{cases} z + w = 3 \\ y - w = -1 \\ x = 1 \end{cases}$$

$$\leadsto \begin{cases} x = 1 \\ y = -1 + t \\ z = 3 - t \\ w = t \end{cases}$$

↑  $t=w$  is a "free variable".

This system has  $\infty$ 'ly many solutions, one for each  $t \in \mathbb{R}$ .